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On factors of g -measures

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In memory of Floris Takens

Abstract

We show that fully supported g -measures on a shift space $A^{\mathbb{Z}_+}$, $|A| < \infty$, remain g -measures under single site renormalization transformations (1-block factors).

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1. Introduction

It is well known that functions (factors) of Markov processes are not necessarily Markov of any finite order. Under very mild conditions, processes which are functions of Markov chains, have *regular conditional probabilities*. More specifically, let $\{X_n\}$ be a stationary finite state Markov chain with a strictly positive transition probability matrix

$$\mathbf{P} = (p_{i,j}) > 0$$

and $Y_n = \pi(X_n)$ be its factor process. Denote by \mathbb{Q} the invariant measure of $\{Y_n\}$, note that $\mathbb{Q} = \mathbb{P} \circ \pi^{-1}$ where \mathbb{P} is the invariant measure of $\{X_n\}$. Then

- the sequence of conditional probabilities $\mathbb{Q}(y_0|y_{-n}^{-1})$ converges uniformly (in $y_{-\infty}^0$) as $n \rightarrow \infty$ to the limit denoted by $\mathbb{Q}(y_0|y_{-\infty}^{-1})$;
- there exist $C > 0$ and $\theta \in (0, 1)$ such that

$$\beta_n := \sup_{y_{-n}^0} \sup_{\tilde{y}_{-\infty}^{-n-1}, \bar{y}_{-\infty}^{-n-1}} \left| \mathbb{Q}(y_0|y_{-n}^{-1}\tilde{y}_{-\infty}^{-n-1}) - \mathbb{Q}(y_0|y_{-n}^{-1}\bar{y}_{-\infty}^{-n-1}) \right| \leq C\theta^n \quad (1.1)$$

for all $n \in \mathbb{N}$.

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Despite the fact that the factor of a Markov chain can have infinite memory, the influence of past values decays exponentially. This important result has been obtained independently in Statistics, Information Theory, and Probability Theory, see [12] for references and discussion of various approaches.

The previous result can also be interpreted as follows: invariant measures of functions of Markov chains are g -measures, see below for the exact definition. The next natural question to address is what are the properties of single-site factors of g -measures, and most importantly, do they remain within the class of g -measures? Recently first results have been obtained in [3,9,10] using various methods. In the present paper we prove that indeed factors of g -measures are again g -measures, using a novel approach.

1.1. Notation

Suppose $X = A^{\mathbb{Z}_+}$ for some finite set A with $|A| > 1$; $T : X \rightarrow X$ is a left shift. Denote by $C(X, \mathbb{R})$ the space of all real-valued continuous functions on X . A g -function on X is a continuous positive function g such that for all $x \in X$,

$$\sum_{x' \in T^{-1}x} g(x') = \sum_{a \in A} g(ax) = 1. \quad (1.2)$$

A probability measure μ on X is called a g -measure if for μ -almost all $x = (x_0, x_1, \dots) \in X$,

$$\mu(x_0|x_1, x_2, \dots) = g(x).$$

Equivalently (see e.g., [13]), μ is a g -measure if for all $f \in C(X, \mathbb{R})$ one has

$$\begin{aligned} \int_X f(x) \mu(dx) &= \int_X \left[\sum_{x' \in T^{-1}(x)} f(x') g(x') \right] \mu(dx) \\ &= \int_X \left[\sum_{a \in A} f(ax) g(ax) \right] \mu(dx). \end{aligned} \quad (1.3)$$

Condition (1.3) can be viewed as a one-sided variant of the so-called Dobrushin–Lanford–Ruelle equations which provide a definition of Gibbs states in Statistical Mechanics. In many cases g -measures are also referred to as Gibbs measures.

For $g \in C(X, \mathbb{R})$ and $n \in \mathbb{Z}_+$ let

$$\text{var}_n(g) = \sup_{x, \tilde{x}: x_0^n = \tilde{x}_0^n} |g(x) - g(\tilde{x})|.$$

Since g is continuous, $\text{var}_n(g) \rightarrow 0$ as $n \rightarrow \infty$. We say that g has *summable variation* if

$$C_g := \sum_{n \geq 0} \text{var}_n(g) < \infty.$$

Walters [13] established that a g -function of summable variation admits a unique g -measure. If g is continuous and positive on X , then $\log g$ is a continuous function on X . It is easy to verify that

$$C_g = \sum_n \text{var}_n(g) < \infty \iff C_{\log g} = \sum_n \text{var}_n(\log g) < \infty.$$

Suppose B is a finite set with $1 < |B| < |A|$, and π is surjective map from A to B . Let $Y = B^{\mathbb{Z}_+}$, and let π denote also the map from X onto Y given by

$$y = \pi(x), \quad \text{where } y_n = \pi(x_n) \quad \forall n \in \mathbb{Z}_+.$$

If μ is a Borel probability measure on X , let $\nu = \pi_*\mu = \mu \circ \pi^{-1}$ be the push-forward of μ :

$$\nu(C) = \mu(\pi^{-1}C) \quad \text{for all Borel } C \subseteq Y \ (C \in \mathfrak{B}(Y)).$$

Theorem 1.1. *Suppose μ is a g -measure on X for some continuous positive function g with summable variation. Then $\nu = \mu \circ \pi^{-1}$ is a \tilde{g} -measure on Y for some continuous positive function \tilde{g} on Y .*

[Theorem 1.1](#) has been obtained earlier by Redig and Wang [10], and under stronger conditions on the decay of variations of g by Chazottes and Ugalde [3], and Kempton and Pollicott [9].

In this paper we provide a proof of [Theorem 1.1](#) using a novel method based on the construction of a *continuous disintegration* of the original measures μ . Our approach relies heavily on the theory of non-homogeneous equilibrium states of Fan and Pollicott [5]. In the last section, we compare the estimates of the decay rates $\text{var}_n(\tilde{g})$ obtained using methods of [3, 9, 10] and of the present paper. It turns out that the present method provides suboptimal bounds. On the other hand, the present approach should be useful in the study of the preservation of the Gibbs property of random fields under renormalization transformations. For random fields, the general picture is much more complex: even Markov random fields may lose the Gibbs property under single-site transformations. Van Enter et al. proposed in [11] a criterion for the loss/preservation of Gibbs property based on the absence of the so-called hidden phase transitions. Though the criterion has not been established in complete generality, all known cases confirm the conjecture. In the last section we will also discuss the relation between the criterion based on the hidden phase transitions and the method of the present paper.

2. Disintegration of measures and criteria for Gibbsianity

For every $y \in Y$ denote by X_y the fiber over y :

$$X_y = \pi^{-1}(y) \subset X.$$

For every y , X_y is a closed, but not necessarily translation invariant subset of X .

Definition 2.1. A family of measures $\mu_Y = \{\mu_y\}_{y \in Y}$ is called a family of *conditional measures* for μ on fibers X_y if

- (a) μ_y is a Borel probability measure on X_y , $\mu_y(X_y) = 1$;
- (b) for all $f \in L^1(X, \mu)$, the map

$$y \rightarrow \int_{X_y} f(x) \mu_y(dx)$$

is measurable and

$$\int_X f(x) \mu(dx) = \int_Y \int_{X_y} f(x) \mu_y(dx) \nu(dy).$$

Every measure μ admits at least one family of conditional measures on fibers (also known as a *disintegration of measure*, [1, Section 1.0.8, p.9]). Note also that

$$\int_{X_y} f(x) \mu_y(dx) = \mathbb{E}_\mu \left(f | \pi^{-1} \mathfrak{B}(Y) \right),$$

where $\pi^{-1} \mathfrak{B}(Y)$ is the σ -algebra of sets

$$\left\{ \pi^{-1}(C) : C \in \mathfrak{B}(Y) \right\}.$$

Theorem 2.2. Suppose μ is a g -measure for some continuous positive function g . Suppose also that $\pi : X \rightarrow Y$ is such that μ admits a family of conditional measures $\mu_Y = \{\mu_y\}_{y \in Y}$ on fibers $\{X_y\}_{y \in Y}$ with an additional property that for every $f \in C(X, \mathbb{R})$ the map

$$y \mapsto \int_{X_y} f(x) \mu_y(dx) \quad (2.1)$$

is continuous on Y (in the product topology). Then $\nu = \mu \circ \pi^{-1}$ is a \tilde{g} -measure on Y , where $\tilde{g} : Y \rightarrow (0, 1)$ is given by

$$\tilde{g}(y) = \tilde{g}((y_0, y_1, y_2, \dots)) = \int_{X_y} \left[\sum_{a \in \pi^{-1}(y_0)} g((a, x_1, x_2, \dots)) \right] \mu_y(dx).$$

Proof. It is sufficient to show that for all $h \in C(Y, \mathbb{R})$ one has

$$\int_Y h(y) \nu(dy) = \int_Y \left[\sum_{b \in B} h(by) \tilde{g}(by) \right] \nu(dy).$$

Suppose $h \in C(Y, \mathbb{R})$. Then

$$\begin{aligned} \int_Y h(y) \nu(dy) &\stackrel{(a)}{=} \int_X (h \circ \pi)(x) \mu(dx) \\ &\stackrel{(b)}{=} \int_X \left[\sum_{a \in A} (h \circ \pi)(ax) g(ax) \right] \mu(dx) \\ &= \int_X \left[\sum_{b \in B} \sum_{a \in \pi^{-1}(b)} (h \circ \pi)(ax) g(ax) \right] \mu(dx) \\ &\stackrel{(c)}{=} \int_Y \int_{X_y} \left[\sum_{b \in B} \sum_{a \in \pi^{-1}(b)} (h \circ \pi)(ax) g(ax) \right] \mu_y(dx) \nu(dy) \\ &\stackrel{(d)}{=} \int_Y \sum_{b \in B} h(by) \left(\int_{X_y} \left[\sum_{a \in \pi^{-1}(b)} g(ax) \right] \mu_y(dx) \right) \nu(dy) \\ &\stackrel{(e)}{=} \int_Y \sum_{b \in B} h(by) \tilde{g}(by) \nu(dy), \end{aligned} \quad (2.2)$$

where equalities (a), (b), (c), (d), and (e) hold because $\nu = \mu \circ \pi^{-1}$, μ is a g -measure, $\{\mu_y\}$ is a system of conditional probabilities for μ on fibers $\{X_y\}$, $(h \circ \pi)(ax) = h(by)$ on fiber X_y for $b = \pi(a)$, and by definition of \tilde{g} , respectively.

It is obvious that \tilde{g} is continuous by (2.1). Moreover, \tilde{g} is strictly positive, as an integral of a strictly positive function. Since g satisfies the normalization condition (1.2), then so does \tilde{g} . \square

Remark 2.1. A family of conditional measures on fibers is essentially unique: if $\{\mu_y\}, \{\tilde{\mu}_y\}$ are two such families, then $\mu_y = \tilde{\mu}_y$ for ν -a.a. y . On the other hand, there is at most one family $\{\mu_y\}$ such that the map

$$y \mapsto \int_{X_y} f(x) \mu_y(dx),$$

is continuous for every $f \in C(X, \mathbb{R})$.

3. Conditional probabilities for g -measures of functions of summable variation

Suppose μ is a g -measure for some positive continuous function g with summable variation. We will use the results of [5] to construct a family of conditional probabilities on fibers $\{X_y\}$ satisfying the condition (2.1).

Fix $y \in Y$, then the fiber X_y ,

$$X_y = \pi^{-1}(y) = \prod_{n=0}^{\infty} \pi^{-1}(y_n)$$

is a *non-homogeneous symbolic space* in the sense of [5]. Recall also the following definition:

Definition 3.1 ([5]). A sequence $G^y = \{g_n^y\}_{n \geq 0}$ of non-negative functions on X_y is called a *sequence of potentials* if for any $n \geq 0$, $g_n^y(x)$ does not depend on the first n -coordinates (x_0, \dots, x_{n-1}) , i.e.,

$$g_n^y(x) = g_n^y(x_n x_{n+1} \dots).$$

Further, it is said to be *normalized* if for any $n \geq 0$

$$\sum_{a \in \pi^{-1} y_n} g_n^y(ax_{n+1}x_{n+2} \dots) = 1.$$

For $n \geq 0$, we denote by G_n^y the product of the first $n + 1$ functions g_k^y , $k = 0, \dots, n$,

$$G_n^y(x) = \prod_{j=0}^n g_j^y(x). \quad (3.1)$$

For a given positive continuous function g on X , and $y \in Y$, we define the sequence of potentials $\{g_n^y(x)\}$ as follows. Firstly, define $G_n^y(x)$ on X_y as

$$G_n^y(x) = \frac{\prod_{k=0}^n g(x_k^n x_{n+1}^{+\infty})}{\sum_{\bar{x}_0^n \in \pi^{-1} y_0^n} \prod_{k=0}^n g(\bar{x}_k^n x_{n+1}^{+\infty})}, \quad (3.2)$$

where we use a shorthand notation x_k^n for $x_k x_{k+1} \dots x_n$ and $x_{n+1}^{+\infty}$ for $x_{n+1} x_{n+2} \dots$, respectively.

Now we wish to define a sequence of potentials $\{g_n^y(x)\}$, such that $\prod_{j=0}^n g_j^y(x) = G_n^y(x)$, with $G_n^y(x)$ given by (3.2). For $n = 0$, one has

$$g_0^y(x) := G_0^y(x) = \frac{g(x_0 x_1^{+\infty})}{\sum_{\bar{x}_0 \in \pi^{-1} y_0} g(\bar{x}_0 x_1^{+\infty})}, \quad (3.3)$$

and for $n \geq 1$,

$$g_n^y(x) := \frac{G_n^y(x)}{G_{n-1}^y(x)} = g(x_n x_{n+1}, \dots) \frac{\sum_{\bar{x}_0^{n-1} \in \pi^{-1} y_0^{n-1}} \prod_{k=0}^{n-1} g(\bar{x}_k^{n-1} x_n x_{n+1}^{+\infty})}{\sum_{\bar{x}_0^n \in \pi^{-1} y_0^n} \prod_{k=0}^n g(\bar{x}_k^n x_{n+1}^{+\infty})}. \quad (3.4)$$

Clearly, $g_n^y(x)$ depends only on x_n, x_{n+1}, \dots , and (3.1) holds automatically. We now have to check that the sequence of potentials $G^y = \{g_n^y\}$ given by (3.3), (3.4) is normalized. Indeed, for $n = 0$ the claim is immediate, and for $n \geq 1$, one has

$$\sum_{z_n \in \pi^{-1} y_n} g_n(z_n x_{n+1}^{+\infty}) = \frac{\sum_{\bar{x}_0^{n-1} \in \pi^{-1} y_0^{n-1}} \sum_{z_n \in \pi^{-1} y_n} \prod_{k=0}^{n-1} g(\bar{x}_k^{n-1} z_n x_{n+1}^{+\infty}) g(z_n x_{n+1}, \dots)}{\sum_{\bar{x}_0^n \in \pi^{-1} y_0^n} \prod_{k=0}^n g(\bar{x}_k^n x_{n+1}^{+\infty})} = 1.$$

In conclusion, for every continuous g we presented a sequence of normalized potentials $\{g_n^y\}$. In order to apply the results of [5], we have to validate a number of technical conditions related to the behavior of families $\{g_n^y\}$ on fibers X_y .

Lemma 3.2. *Suppose g is a positive continuous function on X with summable variation, $\pi : X \rightarrow Y$ is a 1-block factor. For the normalized sequence of potentials $G^y = \{g_n^y\}$ given by (3.3), (3.4), the following properties hold:*

$$g_{\min}^y = \inf \{g_n^y(x) \mid x \in X_y, n \geq 0\} \geq \frac{\inf g}{\sup g} \frac{e^{-C_{\log g}}}{|A|} > 0, \quad (3.5)$$

$$V^y = \sup \left\{ \frac{G_n^y(x)}{G_n^y(\tilde{x})} \mid x, \tilde{x} \in X_y, x_0^n = \tilde{x}_0^n, n \geq 0 \right\} \leq e^{2C_{\log g}} < \infty. \quad (3.6)$$

Proof. For every $m \in \mathbb{Z}_+$ and all $\bar{x}_0^m, x, \tilde{x} \in X$ one has

$$\begin{aligned} \frac{\prod_{k=0}^m g(\bar{x}_k^m x_{m+1}^{+\infty})}{\prod_{k=0}^m g(\bar{x}_k^m \tilde{x}_{m+1}^{+\infty})} &\leq \exp \left(\sum_{k=0}^m |\log g(\bar{x}_k^m x_{m+1}^{+\infty}) - \log g(\bar{x}_k^m \tilde{x}_{m+1}^{+\infty})| \right) \\ &\leq \exp \left(\sum_{k=0}^{+\infty} \text{var}_k(\log g) \right) =: e^{C_{\log g}} < \infty, \end{aligned} \quad (3.7)$$

and therefore, for any $n \geq 0$ and x , one has

$$\begin{aligned} g_n^y(x) &\geq \left(\inf_{\tilde{x} \in X} g(\tilde{x}) \right) \min_{\tilde{x}_0^{n-1} \in A^n} \frac{\prod_{k=0}^{n-1} g(\tilde{x}_k^{n-1} x_n x_{n+1}^{+\infty})}{\sum_{\tilde{x}_n \in \pi^{-1}(y_n)} \prod_{k=0}^{n-1} g(\tilde{x}_k^{n-1} \tilde{x}_n x_{n+1}^{+\infty}) g(\tilde{x}_n x_{n+1}^{+\infty})} \\ &\geq \frac{\inf g}{\sup g} \frac{e^{-C_{\log g}}}{|A|} > 0. \end{aligned}$$

Similarly, for $x, \tilde{x} \in X_y$ with $x_0^n = \tilde{x}_0^n$, one has

$$\frac{G_n^y(x)}{G_n^y(\tilde{x})} \leq \left(\sup_{x, \tilde{x}: x_0^n = \tilde{x}_0^n} \frac{\prod_{k=0}^n g(x_k^n x_{n+1}^{+\infty})}{\prod_{k=0}^n g(x_k^n \tilde{x}_{n+1}^{+\infty})} \right)^2 \leq e^{2C_{\log g}}. \quad \square$$

Lemma 3.3 (Probabilistic Interpretation of Products G_n^y). For any $n \in \mathbb{Z}_+$, $y \in Y$, and $x \in X_y$, the sequence

$$F_{n,k}^y(x) := \frac{\mu(x_0^n | x_{n+1}^{n+k})}{\mu(\pi^{-1}(y_0^n) | x_{n+1}^{n+k})} = \frac{\mu(x_0^n | x_{n+1}^{n+k})}{\sum_{\tilde{x}_0^n \in \pi^{-1} y_0^n} \mu(\tilde{x}_0^n | x_{n+1}^{n+k})}, \quad k \in \mathbb{N},$$

converges to $G_n^y(x)$ as $k \rightarrow \infty$ uniformly (in x, y and n):

$$\sup_{x,y} |G_n^y(x) - F_{n,k}^y(x)| \leq e^{2 \sum_{j=k}^{\infty} \text{var}_j(\log g)} - 1.$$

Proof. If μ is a g -measure for g of summable variation, then $\mu(x_0 | x_1^m) \rightarrow g(x)$ as $m \rightarrow \infty$ uniformly in x . In fact,

$$\sup_x |\log \mu(x_0 | x_1^m) - \log g(x)| \leq \text{var}_m(\log g),$$

and therefore

$$e^{-\text{var}_m(\log g)} \leq \frac{\mu(x_0 | x_1^m)}{g(x)} \leq e^{\text{var}_m(\log g)}$$

for all x . Thus for all $k, n \in \mathbb{N}$, one has

$$\begin{aligned} \frac{\mu(x_0^n | x_{n+1}^{n+k})}{\prod_{m=0}^n g(x_m^{+\infty})} &= \frac{\prod_{m=0}^n \mu(x_m | x_{m+1}^{n+k})}{\prod_{m=0}^n g(x_m^{+\infty})} \leq \exp \left(\sum_{j=k}^{n+k} \text{var}_j(\log g) \right) \\ &\leq \exp \left(\sum_{j=k}^{+\infty} \text{var}_j(\log g) \right) =: e^{I_k}, \end{aligned}$$

with $I_k = \sum_{j=k}^{+\infty} \text{var}_j(\log g) \rightarrow 0$ as $k \rightarrow \infty$. Similarly,

$$\frac{\mu(x_0^n | x_{n+1}^{n+k})}{\prod_{m=0}^n g(x_m^{+\infty})} \geq \exp \left(- \sum_{j=k}^{+\infty} \text{var}_j(\log g) \right) = e^{-I_k}.$$

Finally, one concludes that

$$e^{-2I_k} \leq \frac{F_{n,k}^y(x)}{G_n^y(x)} \leq e^{2I_k}$$

for all n, k, x, y with $x_0^n \in \pi^{-1}y_0^n$. Combining the last inequality with the fact that $G_n^y(x) < 1$, we derive the result. \square

3.1. Non-homogeneous equilibrium states

Define a sequence of averaging operators $P_n^y : C(X_y, \mathbb{R}) \rightarrow C(X_y, \mathbb{R})$, $n \geq 0$, by

$$P_n^y f(x) = \sum_{a_0^n \in \pi^{-1}y_0^n} G_n^y(a_0, \dots, a_n x_{n+1}, \dots) f(a_0, \dots, a_n x_{n+1}, \dots).$$

Operators P_n^y are positive and satisfy $P_n^y \mathbf{1} = \mathbf{1}$ for all $n \in \mathbb{Z}_+$. A probability measure ρ on X_y is called a *non-homogeneous equilibrium state associated to $G^y = \{g_n^y\}$* if $P_n^{y*} \rho = \rho$, i.e.,

$$\int_{X_y} P_n^y f(x) \rho(dx) = \int_{X_y} f(x) \rho(dx)$$

for all $f \in C(X_y, \mathbb{R})$ and every $n \geq 0$.

If g is a positive continuous function of summable variation, then the result of Lemma 3.2 ($g_{\min}^y > 0$ and $V^y < \infty$), allows us to apply Theorem 1 [5, p. 102] to conclude that for every y there is a unique non-homogeneous equilibrium state associated to G^y . We denote this unique equilibrium state by μ_y .

Moreover, again by the same theorem, we conclude that

$$P_n^y f(x) \rightarrow \int_{X_y} f(x) \mu_y(dx)$$

for all continuous functions $f \in C(X_y, \mathbb{R})$. Finally, Theorem 2 [5] establishes that this convergence is uniform in x on X_y : namely,

$$\sup_{x \in X_y} \left| P_n^y f(x) - \int_{X_y} f d\mu_y \right| \leq \text{var}_{n_0}(f) + 2\|f\|_{C(X_y, \mathbb{R})} \left(\sum_{j=1}^k V_{n_{j-1}, n_j}^y + \gamma_y^k \right),$$

for every $k \in \mathbb{N}$ and any choice $0 \leq n_0 < n_1 < \dots < n_k \leq n$, and where

$$\gamma_y = \frac{(V^y)^2 - g_{\min}^y}{(V^y)^2 + g_{\min}^y} < 1,$$

g_{\min}^y , V^y are given by (3.5), (3.6), respectively, and for all $s < t$

$$V_{s,t}^y = \sup \left\{ \frac{G_s^y(x)}{G_s^y(\tilde{x})} : x, \tilde{x} \in X_y, x_0^t = \tilde{x}_0^t \right\} - 1.$$

By arguments similar to that of the proof of Lemma 3.3, we obtain that

$$V_{s,t}^y \leq \left(\sup_{x_0^s \in \pi^{-1}y_0^s} \frac{\prod_{k=0}^s g(\tilde{x}_k^s x_{s+1}^t x_{t+1}^\infty)}{\prod_{k=0}^s g(\tilde{x}_k^s x_{s+1}^t \tilde{x}_{t+1}^\infty)} \right)^2 - 1 \leq \exp \left(2 \sum_{j=t-s}^t \text{var}_j(\log g) \right) - 1.$$

Finally, since all bounds on g_{\min}^y , V^y and $V_{s,t}^y$ are uniform in y , for $f \in C(X, \mathbb{R})$, we obtain the following uniform bound

$$\sup_{y \in Y} \sup_{x \in X_y} \left| P_n^y f - \int_{X_y} f d\mu_y \right| \leq \text{var}_{n_0}(f) + 2\|f\|_{C(X, \mathbb{R})} \left(\sum_{j=1}^k V_{n_{j-1}, n_j} + \gamma^k \right),$$

where

$$\gamma = \frac{|A| \exp(5C_{\log g}) \sup g - \inf g}{|A| \exp(5C_{\log g}) \sup g + \inf g} < 1, \quad V_{s,t} = \exp \left(2 \sum_{j=t-s}^t \text{var}_j(\log g) \right) - 1, \\ \forall s < t,$$

again for any choice $0 \leq n_0 < n_1 \leq \dots < n_k \leq n$.

Let

$$B_n^g(f) = \inf_{k \geq 0, 0 \leq n_0 < n_1 < \dots < n_k \leq n} \left\{ \text{var}_{n_0}(f) + 2\|f\|_{C(X, \mathbb{R})} \left(\sum_{j=1}^k V_{n_{j-1}, n_j} + \gamma^k \right) \right\},$$

and hence

$$\sup_{y \in Y} \sup_{x \in X_y} \left| P_n^y f - \int_{X_y} f d\mu_y \right| \leq B_n^g(f). \quad (3.8)$$

We claim that $B_n^g(f) \rightarrow 0$ as $n \rightarrow \infty$. First of all, observe that since g has summable variation, $\sum_{j=0}^\infty \text{var}_j(g) = C_{\log g} < \infty$, and thus there exists a constant \bar{C} such that for all $s < t$,

$$V_{s,t} = \exp \left(2 \sum_{j=t-s}^t \text{var}_j(\log g) \right) - 1 \leq \bar{C} \sum_{j=t-s}^t \text{var}_j(\log g).$$

Choose some $n_0 \in \mathbb{N}$, and let $n_{j+1} = 2n_j + 1$ for $j = 0, \dots, k-1$, where k is a maximal positive integer so that $n_k \leq n$. Then

$$\sum_{j=1}^k V_{n_{j-1}, n_j} \leq \bar{C} \sum_{j=1}^k \sum_{i=n_{j-1}}^{n_j} \text{var}_i(g).$$

Since $n_{j+1} > 2n_j$ for all j , the intervals $[n_1 - n_0, n_1], [n_2 - n_1, n_2], \dots$, are disjoint, and hence

$$\sum_{j=1}^k V_{n_{j-1}, n_j} \leq \bar{C} \sum_{i=n_0}^\infty \text{var}_i(g).$$

Since $n_j = 2^j n_0 + 2^j - 1$ for all j , k satisfies $2^k(n_0 + 1) \leq n + 1$, and since k was chosen to be maximal, one has

$$k = \left\lceil \log_2 \frac{n+1}{n_0+1} \right\rceil \geq \log_2 \frac{n+1}{n_0+1} - 1 = \log_2 \frac{n+1}{2(n_0+1)}.$$

Thus for some positive constants c_1, c_2 ,

$$B_n^g(f) \leq \text{var}_{n_0}(f) + c_1 \sum_{i=n_0}^{\infty} \text{var}_i(g) + c_2 \left(\frac{n_0+1}{n+1} \right)^{\log_2 \frac{1}{\gamma}}.$$

Since both $\text{var}_{n_0}(f)$ and $\sum_{i=n_0}^{\infty} \text{var}_i(g)$ tend to 0 as $n_0 \rightarrow \infty$, it is evident that we can always choose $n_0 = n_0(n)$, such that the right hand side of the previous inequality tends to 0 as $n \rightarrow \infty$. For example, $n_0 = \sqrt{n}$ suffices. Thus, indeed,

$$B_n^g(f) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now we are ready to proceed with the proof of key lemma establishing the continuity of the family of non-homogeneous equilibrium states $\{\mu_y\}$.

Lemma 3.4. *Suppose g is a positive continuous function with summable variation on X , $\pi : X \rightarrow Y$ is a 1-block factor. For every y , let μ_y be the unique non-homogeneous equilibrium state on X_y associated to the sequence of potentials $G^y = \{g_n^y\}$, given by (3.3)–(3.4). Then for any $f \in C(X, \mathbb{R})$, the map*

$$y \mapsto \int_{X_y} f(x) \mu_y(dx)$$

is continuous on Y .

Proof. Consider $y, \tilde{y} \in Y$ such that $y_0^N = \tilde{y}_0^N$ for some sufficiently large N . Select two points x, \tilde{x} satisfying:

$$x \in X_y, \quad \tilde{x} \in X_{\tilde{y}}, \quad x_0^N = \tilde{x}_0^N.$$

Finally, put $n = [N/2]$. By Eq. (3.8) one has

$$\left| \int_{X_y} f(x) \mu_y(dx) - \int_{X_{\tilde{y}}} f(x) \mu_{\tilde{y}}(dx) \right| \leq 2B_n^g(f) + \left| P_n^y f(x) - P_n^{\tilde{y}} f(\tilde{x}) \right|.$$

To estimate the last term we proceed as follows:

$$\begin{aligned} & |P_n^y f(x) - P_n^{\tilde{y}} f(\tilde{x})| \\ & \leq \left| \sum_{a_0^n \in \pi^{-1} y_0^n} G_n^y(a_0^n x_{n+1}^\infty) f(a_0^n x_{n+1}^\infty) - \sum_{a_0^n \in \pi^{-1} y_0^n} G_n^y(a_0^n x_{n+1}^\infty) f(a_0^n \tilde{x}_{n+1}^\infty) \right| \\ & \quad + \left| \sum_{a_0^n \in \pi^{-1} y_0^n} G_n^y(a_0^n x_{n+1}^\infty) f(a_0^n \tilde{x}_{n+1}^\infty) - \sum_{a_0^n \in \pi^{-1} y_0^n} G_n^{\tilde{y}}(a_0^n \tilde{x}_{n+1}^\infty) f(a_0^n \tilde{x}_{n+1}^\infty) \right| \\ & \leq \sum_{a_0^n \in \pi^{-1} y_0^n} G_n^y(a_0^n x_{n+1}^\infty) |f(a_0^n x_{n+1}^\infty) - f(a_0^n \tilde{x}_{n+1}^\infty)| \end{aligned}$$

$$\begin{aligned}
& + \|f\|_{C(X, \mathbb{R})} \sum_{a_0^n \in \pi^{-1}y_0^n} G_n^{\tilde{y}}(a_0^n \tilde{x}_{n+1}^\infty) \left| \frac{G_n^y(a_0^n x_{n+1}^\infty)}{G_n^{\tilde{y}}(a_0^n \tilde{x}_{n+1}^\infty)} - 1 \right| \\
& \leq \text{var}_N(f) + \|f\|_{C(X, \mathbb{R})} V_{n,N},
\end{aligned}$$

where we have used that

$$\left| \frac{G_n^y(a_0^n x_{n+1}^\infty)}{G_n^{\tilde{y}}(a_0^n \tilde{x}_{n+1}^\infty)} - 1 \right| \leq \sup_{a_0^n, x_{n+1}^N, x_{N+1}^\infty, \tilde{x}_{N+1}^\infty} \left\{ \frac{G_n^y(a_0^n x_{n+1}^N x_{N+1}^\infty)}{G_n^{\tilde{y}}(a_0^n x_{n+1}^N \tilde{x}_{N+1}^\infty)} \right\} - 1 \leq V_{n,N}.$$

Hence, we conclude that for y, \tilde{y} with $y_0^N = \tilde{y}_0^N$

$$\left| \int_{X_y} f(x) \mu_y(dx) - \int_{X_{\tilde{y}}} f(x) \mu_{\tilde{y}}(dx) \right| = o(1) \quad \text{as } N \rightarrow \infty,$$

and hence the map $y \rightarrow \int_{X_y} f d\mu_y$ is continuous. \square

3.2. Disintegration into equilibrium states

Now we are ready to show that if $\{\mu_y\}$ is a family of non-homogeneous equilibrium states associated to potentials $\{g_n^y\}$, then

$$\int_X f(x) \mu(dx) = \int_Y \int_{X_y} f(x) \mu_y(dx) \nu(dy) \quad (3.9)$$

for all $f \in C(X, \mathbb{R})$. It is sufficient to check (3.9) for indicator functions of cylinders. Suppose $a_0^m \in A^{m+1}$ and let $f(x) = \mathbb{I}_{[a_0^m]}(x) = 1$ if $x_0^m = a_0^m$ and 0 otherwise. Suppose $n \geq m$, then for all $y \in Y$ with $y_0^m = \pi(a_0^m)$ we have

$$P_n^y f(x) = \sum_{\tilde{a}_0^n \in \pi^{-1}y_0^n} G_n^y(\tilde{a}_0^n x_{n+1}^\infty) f(\tilde{a}_0^n x_{n+1}^\infty) = \sum_{\tilde{a}_{m+1}^n \in \pi^{-1}y_{m+1}^n} G_n^y(a_0^m \tilde{a}_{m+1}^n x_{n+1}^\infty).$$

Suppose $n \in \mathbb{N}$ is sufficiently large, and let $t = 2n$. Then

$$I_f := \int_Y \nu(dy) \int_{X_y} f(x) \mu_y(dx) = \sum_{b_0^t \in B^{t+1}} \nu(b_0^t) \sup_{y \in [b_0^t]} \int_{X_y} f(x) \mu_y(dx) + \alpha_n, \quad (3.10)$$

where by Lemma 3.4

$$|\alpha_n| \leq 2B_n^g(f) + \text{var}_{2n}(f) + \|f\|_{C(X, \mathbb{R})} V_{n,2n} = o(1) \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$\sum_{b_0^t \in B^{t+1}} \nu(b_0^t) \sup_{y \in [b_0^t]} \int_{X_y} f(x) \mu_y(dx) = \sum_{b_0^t \in B^{t+1}} \nu(b_0^t) \sup_{y \in [b_0^t]} \sup_{x \in X_y} P_n^y f(x) + \beta_n,$$

where by (3.8)

$$|\beta_n| \leq B_n^g(f) = o(1) \quad \text{as } n \rightarrow \infty.$$

Hence, we conclude that

$$I_f = \sum_{b_0^t \in B^{t+1}} \nu(b_0^t) \sup_{y \in [b_0^t]} \sup_{x \in X_y} \left[\mathbb{I}_{b_0^m = \pi(a_0^m)} \sum_{\bar{a}_{m+1}^n \in \pi^{-1}y_{m+1}^n} F_{n,n}^y(a_0^m \bar{a}_{m+1}^n x_{n+1}^{+\infty}) \right] + \alpha_n + \beta_n + \gamma_n \quad (3.11)$$

where by Lemma 3.3

$$|\gamma_n| \leq \sup_{y \in Y} \sup_{x \in X_y} |G_n^y(x_0^n x_{n+1}^\infty) - F_{n,n}^y(x_0^n x_{n+1}^\infty)| \leq e^{2 \sum_{j=n}^\infty \text{var}_j(\log g)} - 1 = o(1) \quad \text{as } n \rightarrow \infty.$$

Let $\delta_n = \alpha_n + \beta_n + \gamma_n = o(1)$ as $n \rightarrow \infty$. Taking into account that $F_{n,n}^y(x)$ depends only on x_0^{2n} , we can continue (3.11) as follows:

$$I_f = \sum_{b_{m+1}^{2n} \in B^{2n-m}} \nu\left(\pi(a_0^m) b_{m+1}^{2n}\right) \sup_{x_{n+1}^{2n} \in \pi^{-1} b_{m+1}^{2n}} \frac{\mu(a_0^m \pi^{-1}(b_{m+1}^n) | x_{n+1}^{2n})}{\mu(\pi^{-1}(\pi a_0^m) \pi^{-1}(b_{m+1}^n) | x_{n+1}^{2n})} + \delta_n. \quad (3.12)$$

Repeating now the argument (3.10)–(3.12) but now for the infimum, we similarly conclude that

$$I_f = \sum_{b_{m+1}^{2n} \in B^{2n-m}} \nu\left(\pi(a_0^m) b_{m+1}^{2n}\right) \inf_{x_{n+1}^{2n} \in \pi^{-1} b_{m+1}^{2n}} \frac{\mu(a_0^m \pi^{-1}(b_{m+1}^n) | x_{n+1}^{2n})}{\mu(\pi^{-1}(\pi a_0^m) \pi^{-1}(b_{m+1}^n) | x_{n+1}^{2n})} + \tilde{\delta}_n, \quad (3.13)$$

where again $\tilde{\delta}_n = o(1)$.

Finally, since

$$\min_i \frac{a_i}{b_i} \leq \frac{\sum_i c_i a_i}{\sum_i c_i b_i} \leq \max_i \frac{a_i}{b_i}$$

for non-negative sequences $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, we conclude that

$$\left| I_f - \sum_{b_{m+1}^{2n} \in B^{2n-m}} \nu\left(\pi(a_0^m) b_{m+1}^{2n}\right) \frac{\sum_{x_{n+1}^{2n} \in \pi^{-1} b_{m+1}^{2n}} \mu(x_{n+1}^{2n}) \mu(a_0^m \pi^{-1}(b_{m+1}^n) | x_{n+1}^{2n})}{\sum_{x_{n+1}^{2n} \in \pi^{-1} b_{m+1}^{2n}} \mu(x_{n+1}^{2n}) \mu(\pi^{-1}(\pi a_0^m) \pi^{-1}(b_{m+1}^n) | x_{n+1}^{2n})} \right| = o(1).$$

On the other hand, the last sum equals

$$\begin{aligned} & \sum_{b_{m+1}^{2n} \in B^{2n-m}} \nu\left(\pi(a_0^m) b_{m+1}^{2n}\right) \frac{\mu(a_0^m \pi^{-1}(b_{m+1}^n))}{\mu(\pi^{-1}(\pi a_0^m) b_{m+1}^{2n})} \\ &= \sum_{b_{m+1}^{2n} \in B^{2n-m}} \nu\left(\pi(a_0^m) b_{m+1}^{2n}\right) \frac{\mu(a_0^m \pi^{-1}(b_{m+1}^n))}{\nu(\pi(a_0^m) b_{m+1}^{2n})} \end{aligned}$$

$$\begin{aligned}
&= \sum_{b_{m+1}^{2n} \in B^{2n-m}} \mu \left(a_0^m \pi^{-1}(b_{m+1}^{2n}) \right) \\
&= \mu(a_0^m) = \int_X f(x) \mu(dx).
\end{aligned}$$

Therefore, we have obtained the following result

Lemma 3.5. *If for every $y \in Y$, μ_y is a non-homogeneous equilibrium state on X_y associated to G^y , then for any $f \in C(X, \mathbb{R})$*

$$\int_X f(x) \mu(dx) = \int_Y \int_{X_y} f(x) \mu_y(dx),$$

and hence $\{\mu_y\}$ is the family of conditional probabilities for μ on fibers X_y .

Proof of Theorem 1.1. Theorem 1.1 is an immediate corollary of Lemmas 3.5 and 3.4 and Theorem 2.2. \square

4. Conclusions and final remarks

(a) Questions related to the preservation of the Gibbs property under single-site renormalization transformations of random fields have been considered extensively in Statistical Mechanics. As already mentioned in the introduction, for random fields the picture is much more complex. The behavior of equilibrium (Gibbs) states on fibers have been brought in connection to the Gibbs properties of the renormalized measures. The scenario for the loss of Gibbsianity has been proposed by van Enter–Fernandez–Sokal in [11]: *loss of Gibbsianity occurs when there is a hidden phase transition in the original system conditioned on image spins*. Theorem 2.2 remains true in case of Gibbs measures on multidimensional lattices \mathbb{Z}^d . It will be very interesting to understand the exact relation between the continuity of the family of conditional probabilities on fibers and phase transitions in the families of Gibbs measures on fibers (hidden phase transition of [11]).

(b) As we have seen, $\nu = \mu \circ \pi^{-1}$ is a \tilde{g} -measure for some continuous function \tilde{g} . Hence, $\text{var}_n(\tilde{g}) \rightarrow 0$ as $n \rightarrow \infty$. An interesting problem is to express the decay rate of $\text{var}_n(\tilde{g})$ in terms of the decay rate of $\text{var}_n(g)$. Let us recall the results of [3,9,10]. In [3,9], the result of Theorem 2.2 was obtained under stronger conditions on the decay rate of $\text{var}_n(g)$: namely, the authors of [3] require $n^{2+\epsilon} \cdot \text{var}_n(g)$ to be summable for some $\epsilon > 0$, and in [9], the summability of $n \cdot \text{var}_n(g)$ is required. In [10] the result is established under the condition of summability of $\text{var}_n(g)$. Proofs in [3,9,10] employ different techniques.

Moreover, the estimates on the decay rate of $\text{var}_n(g)$ also vary. For Hölder continuous g (i.e., $\text{var}_n(g) = \mathcal{O}(e^{-an})$), [3,9] established a stretched exponential decay for $\text{var}_n(g)$:

$$\text{var}_n(\tilde{g}) = \mathcal{O}(e^{-\beta\sqrt{n}}),$$

while exponential decay

$$\text{var}_n(\tilde{g}) = \mathcal{O}(e^{-\beta n})$$

was found in [10], but also in an earlier paper [4].

In [3,10] it was established that for g with polynomial decay, i.e., $\text{var}_n(g) = \mathcal{O}(n^{-\alpha})$, $\alpha > 0$ is large enough,

$$\text{var}_n(\tilde{g}) = \mathcal{O}(n^{-\alpha+2}),$$

while in [9] it was shown that

$$\text{var}_n(\tilde{g}) = \mathcal{O}(n^{-\alpha+1}).$$

The method of the present paper is less explicit in terms of identification of \tilde{g} than the methods used in [3,9,10]. Nevertheless, it is possible to derive some bounds on the decay rate of $\text{var}_n(\tilde{g})$. Using the expression for \tilde{g} in Theorem 2.2

$$\tilde{g}(y) = \int_{X_y} \left[\sum_{\tilde{x}_0 \in \pi^{-1}y_0} g(\tilde{x}_0 x_1 x_2, \dots) \right] \mu_y(dx),$$

one can relate $\text{var}_n(\tilde{g})$ with the modulus of continuity of the map

$$y \mapsto \int_{X_y} g(x) \mu_y(dx).$$

Lemma 3.4 provides the following bound: for $y, \bar{y} \in Y$ with $y_0^{2n} = \bar{y}_0^{2n}$,

$$\left| \int_{X_y} g(x) \mu_y(dx) - \int_{X_{\bar{y}}} g(x) \mu_{\bar{y}}(dx) \right| \leq B_n^g(g) + \text{var}_n(g) + \|g\| V_{n,2n}.$$

From this expression it is evident that the estimate of $\text{var}_n(\tilde{g})$ cannot be better than $V_{n,2n}$, and, e.g., $V_{n,2n} = \mathcal{O}(n^{-\alpha+1})$ for g with $\text{var}_n(g) = \mathcal{O}(n^{-\alpha})$. Note that by the result of [9], $\text{var}_n(\tilde{g}) = \mathcal{O}(n^{-\alpha+1})$. Nevertheless, in case of Hölder continuous g , similarly to [3,9], we are able to derive a stretched exponential bound on $\text{var}_n(\tilde{g})$, which by the result of [4,10] is suboptimal. Moreover, it is also clear that a better bound cannot be derived with our method.

It is worth mentioning that there is definitely some room for improvement. Consider the following situation: suppose $\pi_1 : X \mapsto Y$, and $\pi_2 : Y \mapsto Z$ are 1-block factor maps. Then $\pi = \pi_2 \circ \pi_1 : X \rightarrow Z$ is still a 1-block factor map, and hence, for g with summable variation and the corresponding g -measure μ , the measures $\lambda = \mu \circ \pi_1^{-1}$ and $\nu = \mu \circ \pi^{-1}$ are g -measures. However, the current results cannot always be applied to show that $\nu = \lambda \circ \pi_2^{-1}$ is a g -measure, because the estimates do not guarantee that the g -function for measure λ has summable variation. At the present moment, we only know that the class of Hölder continuous g -functions closed under taking 1-block factors.

(c) In connection to the previous remark, a natural question is to identify the class of g -functions closed under taking 1-block factors. A good candidate is the class of functions with *square summable variations*:

$$\sum_n \text{var}_n^2(g) < \infty.$$

Uniqueness of g -measures for g with square summable variations has been established in [6]. However, very little is known about the convergence rate of Ruelle–Perron–Frobenius operators in this case, c.f., [7, Question 2, p. 1149]. Nevertheless, we conjecture that this class is indeed closed. The scenario for the loss of Gibbsianity of van Enter–Fernandez–Sokal discussed above, strongly suggests that this is the case. A plausible first step would be the extension of the results of Fan and Pollicott [5] to g -functions of square summable variations.

(d) Another interesting question is to consider properties of factors of g -measures supported on subshifts of finite type. In the past, this question has been considered by Chazottes and

Ugalde [2], who identified a set of conditions sufficient to ensure the g -property of factor measures. Recently, Kempton [8] and Yoo [14] found weaker sufficient conditions.

The approach proposed in the present paper potentially can be extended to address factors of g -measures supported by subshifts of finite type. The theory of non-homogeneous equilibrium states developed in [5] is suitable for treatment of g -measures concentrated on non-homogeneous subshifts of finite type. Furthermore, one has to require that each fiber X_y is a non-homogeneous subshift of finite type. Tom Kempton has kindly informed us that under the conditions of [8], this is indeed the case. Similarly to the present case, and possibly, under additional conditions, the only remaining obstacle is to show the continuity of the family of conditional measures $\{\mu_y\}$.

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